Available at www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
J. Math. Anal. Appl. 286 (2003) 363–367
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Note

A note on Nasr's and Wong's papers

Yuan Gong Sun

Department of Mathematics, Qufu Normal University, Qufu, Shandong 273165, China

Received 14 March 2003
Submitted by J. Wong

Abstract

In the case of oscillatory potentials, we give sufficient conditions for the oscillation of the forced nonlinear second order differential equations with delayed argument in the form

$$x''(t) + q(t)|x(\tau(t))|^{\gamma} \operatorname{sgn} x(\tau(t)) = f(t)$$

in the linear ($\gamma = 1$) and the superlinear ($\gamma > 1$) cases.

Keywords: Oscillation; Forced nonlinear equations; Delayed argument

1. Introduction

Consider the forced nonlinear second order differential equations with delayed argument

$$x''(t) + q(t)\big|x\big(\tau(t)\big)\big|^{\gamma} \operatorname{sgn} x\big(\tau(t)\big) = f(t), \tag{1}$$

where $t \in R_+ = [0, \infty)$, q, τ, f are continuous functions, $\tau(t)$ is nondecreasing, $\tau(t) \leqslant t$ for $t \in R_+$, and $\lim_{t\to\infty} \tau(t) = \infty$. Our interest is to establish oscillation criteria for Eq. (1) that do not assume that q and f be of definite sign. A solution of Eq. (1) is said to be oscillatory if it is defined on some ray $[\alpha, \infty)$ and has unbounded zeros. Eq. (1) is called oscillatory if all its solutions on some ray are oscillatory.

When $\tau(t) = t$, Eq. (1) takes the form

$$x''(t) + q(t)|x(t)|^{\gamma} \operatorname{sgn} x(t) = f(t), \tag{2}$$

E-mail address: yugsun@263.net.

0022-247X/$ – see front matter
doi:10.1016/S0022-247X(03)00460-8

which is the forced Emden–Fowler equation. The oscillatory behavior of Eq. (2) has been studied by many authors (see [6–10] and references therein). In these articles, it is always assumed that $q(t)$ is nonnegative. In this case, one can usually establish oscillation criteria for a more general nonlinear equations by employing a technique introduced by Kartsatos [1,2] where it is additionally assumed that f is the second derivative of an oscillatory function h. This approach has been expressed in [11,12].

Recently, without such a restriction on f and in the case of oscillatory potentials $q(t)$ and $f(t)$, El-Sayed [3], Nasr [4], and Wong [5] studied the oscillation of Eq. (2) with $\gamma = 1$ and Eq. (2) with $\gamma > 1$, respectively. However, it seems to us that nothing has been known about the oscillation of Eq. (1) when the potentials $q(t)$ and $f(t)$ are oscillatory. In the following section, we will establish oscillation criteria for Eq. (1) that do not assume that q and f be of definite sign. Our results extend and improve the main results of El-Sayed [3], Nasr [4], and Wong [5].

2. Main results

Theorem 1. *Suppose that for any $T \geqslant 0$, there exist constants a_1, b_1, a_2, b_2 such that $T \leqslant a_1 < b_1$, $T \leqslant a_2 < b_2$, and*

$$\begin{aligned} &q(t) \geqslant 0 \quad \text{for } t \in \big[\tau(a_1), b_1\big] \cup \big[\tau(a_2), b_2\big], \\ &f(t) \begin{cases} \leqslant 0, & t \in [\tau(a_1), b_1], \\ \geqslant 0, & t \in [\tau(a_2), b_2]. \end{cases} \end{aligned} \tag{3}$$

Denote $D(a_i, b_i) = \{u \in C^1[a_i, b_i],\ u(t) \not\equiv 0,\ u(a_i) = u(b_i) = 0\}$, $i = 1, 2$. If there exists $u \in D(a_i, b_i)$ such that

$$\int_{a_i}^{b_i} \left[u^2(t) q(t) \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} - \big(u'(t)\big)^2\right] dt \geqslant 0, \tag{4}$$

then Eq. (1) *with $\gamma = 1$ is oscillatory.*

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution which is eventually positive. Say $x(\tau(t)) > 0$ when $t \geqslant t_0$ for some t_0 depending on the solution $x(t)$. Denote $w(t) = -x'(t)/x(t)$ for $t \geqslant t_0$. It follows form Eq. (1) that $w(t)$ satisfies the following nonlinear equation:

$$w'(t) = w^2(t) + q(t) \frac{x(\tau(t))}{x(t)} - \frac{f(t)}{x(t)}. \tag{5}$$

By the assumption, we can choose $a_1, b_1 \geqslant t_0$ such that $b > \tau(a_1)$, $\tau^2(a_1) = \tau(\tau(a_1)) \geqslant t_0$, $q(t) \geqslant 0$ for $t \in [\tau(a_1), b_1]$, and $f(t) \leqslant 0$ for $t \in [\tau(a_1), b_1]$. From Eq. (1) we can easily obtain that $x''(t) \leqslant 0$ for $t \in [\tau(a_1), b_1]$. Therefore, we have that for $t \in [\tau(a_1), b_1]$,

$$x(t) - x\big(\tau(a_1)\big) = x'(s)\big(t - \tau(a_1)\big) \geqslant x'(t)\big(t - \tau(a_1)\big), \tag{6}$$

where $s \in [\tau(a_1), b_1]$. Noting that $x(t) > 0$ for $t \geqslant \tau(a_1)$, we get by (6) that

$$x(t) \geqslant x'(t)\big(t - \tau(a_1)\big), \quad t \in \big[\tau(a_1), b_1\big],$$

i.e.,

$$\frac{x'(t)}{x(t)} \leqslant \frac{1}{t-\tau(a_1)}, \quad t \in \bigl[\tau(a_1), b_1\bigr]. \tag{7}$$

Integrating (7) from $\tau(t)$ to $t > a$, we obtain

$$\frac{x(\tau(t))}{x(t)} \geqslant \frac{\tau(t)-\tau(a_1)}{t-\tau(a_1)}, \quad t \in (a_1, b_1]. \tag{8}$$

By (5) and (8), we have that for $t \in (a_1, b_1]$,

$$w'(t) \geqslant w^2(t) + q(t)\frac{\tau(t)-\tau(a_1)}{t-\tau(a_1)}. \tag{9}$$

Let $u(t) \in D(a_1, b_1)$ be given as in the hypothesis. Multiplying $u^2(t)$ through (9) and integrating it from $s_1 > a_1$ to b_1, we find

$$\begin{aligned}\int_{s_1}^{b_1} u^2(t)q(t)\frac{\tau(t)-\tau(a_1)}{t-\tau(a_1)}\,dt &\leqslant -\int_{s_1}^{b_1} 2u(t)u'(t)w(t)\,dt - \int_{s_1}^{b_1} u^2(t)w^2(t)\,dt \\ &= \int_{s_1}^{b_1} \bigl(u'(t)\bigr)^2\,dt - \int_{s_1}^{b_1} \bigl[u(t)w(t)+u'(t)\bigr]^2\,dt.\end{aligned}$$

Letting $s_1 \to a_1$ in the above inequality, we get

$$\int_{a_1}^{b_1} u^2(t)q(t)\frac{\tau(t)-\tau(a_1)}{t-\tau(a_1)}\,dt \leqslant \int_{a_1}^{b_1} \bigl(u'(t)\bigr)^2\,dt - \int_{a_1}^{b_1} \bigl[u(t)w(t)+u'(t)\bigr]^2\,dt. \tag{10}$$

We say that

$$\int_{a_1}^{b_1} \bigl[u(t)w(t)+u'(t)\bigr]^2\,dt > 0. \tag{11}$$

Otherwise, we have $u(t)w(t) + u'(t) = 0$ for $t \in [a_1, b_1]$. From the definition of $w(t)$, we see that $x(t)$ is a multiple of $u(t)$, i.e., $x(t)$ has zeros at the two points a_1 and b_1 which is a contradiction to the fact that $x(t) > 0$ on $[a_1, b_1]$. Applying (11) to (10), we obtain that

$$\int_{a_i}^{b_i} \left[u^2(t)q(t)\frac{\tau(t)-\tau(a_i)}{t-\tau(a_i)} - \bigl(u'(t)\bigr)^2\right] dt < 0,$$

which contradicts assumption (4).

In the case of $x(t) < 0$ for $t \geqslant t_0$, we use the function $y(t) = -x(t)$ as a positive solution of the differential equation $x''(t) + q(t)x(\tau(t)) = -f(t)$ and repeat the above procedure on the interval $[a_2, b_2]$ in place of $[a_1, b_1]$. This completes the proof of Theorem 1. $\square$

Remark 1. When $\tau(t) = t$, it is easy to see that Theorem 1 reduces to Theorem 1 of Wong [5] with $p(t) \equiv 1$.

Theorem 2. *Suppose that for any $T \geqslant 0$, there exist constants a_1, b_1, a_2, b_2 such that $T \leqslant a_1 < b_1$, $T \leqslant a_2 < b_2$, and* (3) *holds. If there exists $u \in D(a_i, b_i)$, such that*

$$\int_{a_i}^{b_i} \left[\theta u^2(t) q^{1/\gamma}(t) \left| f(t) \right|^{1-1/\gamma} \frac{\tau(t) - \tau(a_i)}{t - \tau(a_i)} - \left(u'(t)\right)^2 \right] dt \geqslant 0, \tag{12}$$

where $D(a_i, b_i)$ is the same as in Theorem 1 *and $\theta = \gamma(\gamma - 1)^{1/\gamma - 1}$, then Eq.* (1) *with $\gamma > 1$ is oscillatory.*

Proof. Assume, to the contrary, that $x(t)$ is a nonoscillatory solution which is eventually positive. Say $x(\tau(t)) > 0$ when $t \geqslant t_0$ for some t_0 depending on the solution $x(t)$. We may say

$$F(x) = Ax^\gamma - \gamma(\gamma - 1)^{1/\gamma - 1} A^{1/\gamma} B^{1-1/\gamma} x \geqslant -B \quad \text{for } x > 0, \tag{13}$$

where A, B are nonnegative constants. In fact, if $A = 0$, (13) is obvious. On the other hand, if $A > 0$, it is easy to see that $F(x)$ obtains its minimum at $x = ((\gamma - 1)^{1/\gamma - 1} \times A^{1/\gamma - 1} B^{1-1/\gamma})^{1/(\gamma - 1)}$ and $F_{\min} = -B$. Thus, (13) holds. By the assumption, we can choose $a_1, b_1 \geqslant t_0$ such that $b > \tau(a_1)$, $\tau^2(a_1) = \tau(\tau(a_1)) \geqslant t_0$, $q(t) \geqslant 0$ for $t \in [\tau(a_1), b_1]$, and $f(t) \leqslant 0$ for $t \in [\tau(a_1), b_1]$. If we choose $A = q(t)$ and $B = -f(t)$, from (13), we have that for $t \in [\tau(a_1), b_1]$,

$$q(t) x^\gamma\left(\tau(t)\right) - f(t) \geqslant \gamma(\gamma - 1)^{1/\gamma - 1} q^{1/\gamma}(t) \left| f(t) \right|^{1-1/\gamma} x\left(\tau(t)\right). \tag{14}$$

By Eqs. (1) and (14) we obtain

$$x''(t) + \theta q^{1/\gamma}(t) \left| f(t) \right|^{1-1/\gamma} x\left(\tau(t)\right) \leqslant 0. \tag{15}$$

Define $w(t)$ as in Theorem 1. It follows from (15) that $w(t)$ satisfies the following inequality:

$$w'(t) \geqslant w^2(t) + \theta q^{1/\gamma}(t) \left| f(t) \right|^{1-1/\gamma} \frac{x(\tau(t))}{x(t)}.$$

The following proof is similar to that of Theorem 1, and hence is omitted. This completes the proof of Theorem 2. □

Remark 2. When $\tau(t) = t$, it is easy to see that Theorem 2 improves the main result of Nasr [4] since $\theta > 1$.

Now, let us consider an example.

Example. Consider the following differential equation:

$$x''(t) + m \sin t \left| x(t - \pi/4) \right|^\gamma \operatorname{sgn} x(t - \pi/4) = \cos t, \quad \gamma \geqslant 1, \; t \geqslant 0, \tag{16}$$

where $m \geqslant 0$ is a constant, $q(t) = m \sin t$, $\tau(t) = t - \pi/4$, $f(t) = \cos t$. For any $T \geqslant 0$, if we choose $a_1 = 2n\pi + 3\pi/4$, $b_1 = 2n\pi + \pi$, $a_2 = 2n\pi + \pi/4$, $b_1 = 2n\pi + \pi/2$ such that $a_1 \geqslant T$ for sufficiently large integer n, then we have $q(t) \geqslant 0$ for $t \in [\tau(a_1), b_1] \cup [\tau(a_2), b_2]$, $f(t) \leqslant 0$ for $t \in [\tau(a_1), b_1]$, and $f(t) \geqslant 0$ for $t \in [\tau(a_2), b_2]$. Choose $u(t) = \sin 2t \cos 2t$, then $u(t) \in D(a_i, b_i)$, $i = 1, 2$. Noting that for $i = 1, 2$,

$$\int_{a_i}^{b_i} \left(u'(t)\right)^2 dt = 4\int_{a_i}^{b_i} \cos^2 4t = \frac{\pi}{2}.$$

By Theorems 1 and 2, we obtain that if

$$\int_{a_i}^{b_i} u^2(t)q(t)\frac{\tau(t)-\tau(a_i)}{t-\tau(a_i)} = m\int_{a_i}^{b_i} \sin^2 2t\cos^2 2t\sin t\frac{t-a_i}{t-a_i+\pi/4} \geqslant \frac{\pi}{2}$$

and

$$\begin{aligned} &m\theta\int_{a_i}^{b_i} u^2(t)q^{1/\gamma}(t)\left|f(t)\right|^{1-1/\gamma}\frac{\tau(t)-\tau(a_i)}{t-\tau(a_i)} \\ &\quad = m\theta\int_{a_i}^{b_i} \sin^2 2t\cos^2 2t\sin^{1/\gamma} t\left|\cos(t)\right|^{1-1/\gamma} dt \geqslant \frac{\pi}{2}, \quad \gamma > 1, \end{aligned}$$

hold for sufficiently large m, where $\theta = \gamma(\gamma-1)^{1/\gamma-1}$, then Eq. (16) is oscillatory.

Remark 3. In the case of oscillatory potentials $q(t)$ and $f(t)$, we will further the investigation for the oscillation of Eq. (1) with $0 < \gamma < 1$ in our next paper.

References

[1] A.G. Kartsatos, On the maintenance of oscillation of n-th order equations under the effect of a small forcing term, J. Differential Equations 10 (1971) 355–363.

[2] A.G. Kartsatos, Maintenance of oscillations under the effect of a periodic forcing term, Proc. Amer. Math. Soc. 33 (1972) 377–382.

[3] M.A. El-Sayed, An oscillation criterion for a forced second order linear differential equation, Proc. Amer. Math. Soc. 118 (1993) 813–817.

[4] A.H. Nasr, Sufficient conditions for the oscillation of forced super-linear second order differential equations with oscillatory potential, Proc. Amer. Math. Soc. 126 (1998) 123–125.

[5] J.S.W. Wong, Oscillation criteria for a forced second-order linear differential equation, J. Math. Anal. Appl. 231 (1999) 235–240.

[6] M.S. Keener, Solutions of a certain linear nonhomogeneous second order differential equations, Appl. Anal. 1 (1971) 57–63.

[7] H. Teufel, Forced second order nonlinear oscillations, J. Math. Anal. Appl. 40 (1972) 148–152.

[8] A. Skidmore, W. Leighton, On the differential equation $y'' + p(x)y = f(x)$, J. Math. Anal. Appl. 43 (1973) 46–55.

[9] A. Skidmore, J.J. Bowers, Oscillatory behavior of solutions of $y'' + p(x)y = f(x)$, J. Math. Anal. Appl. 49 (1975) 317–323.

[10] S.M. Rainikin, Oscillation theorems for second order nonhomogeneous linear differential equations, J. Math. Anal. Appl. 53 (1976) 550–553.

[11] J.S.W. Wong, Second order nonlinear forced oscillations, SIAM J. Math. Anal. 19 (1988) 667–675.

[12] A.H. Nasr, Necessary and sufficient conditions for the oscillation of forced nonlinear second order differential equations with delayed argument, J. Math. Anal. Appl. 212 (1997) 51–59.